Cellularity of Diagram Algebras as Twisted Semigroup Algebras

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Abstract

The Temperley-Lieb and Brauer algebras and their cyclotomic analogues, as well as the partition algebra, are all examples of twisted semigroup algebras. We prove a general theorem about the cellularity of twisted semigroup algebras of regular semigroups. This theorem, which generalises a recent result of East about semigroup algebras of inverse semigroups, allows us to easily reproduce the cellularity of these algebras.

Key words: cellular, twisted semigroup algebra

1 Introduction

There has been much interest in algebras which have a basis consisting of diagrams, which are multiplied in some natural diagrammatic way. Examples of these so-called *diagram algebras* include the partition, Brauer and Temperley-Lieb algebras. These three examples have been studied extensively in the literature. In particular each has been shown to be *cellular*; this property, introduced by Graham and Lehrer in [5], allows us to easily derive information about the semisimplicity of the algebra and about its representation theory, even in the non-semisimple case.

In the three algebras mentioned above, the product of two diagram basis elements is always a scalar multiple of another basis element. Motivated by

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this observation, we realise these algebras as *twisted semigroup algebras*. We can then reproduce the above cellularity results by proving a general theorem about twisted semigroup algebras, which extends a recent result of East [3].

2 Semigroups

Central to the study of any semigroup are certain relations defined by Green [6], which we now briefly recall. Let S be a semigroup. Write $x \leq_{\mathcal{R}} y$, $x \leq_{\mathcal{L}} y$ or $x \leq_{\mathcal{J}} y$ if x can be obtained from y by, respectively, left multiplication, right multiplication or simultaneous left and right multiplication. Green's relations are the equivalence relations defined by

$$\mathcal{R} = \leq_{\mathcal{R}} \cap \geq_{\mathcal{R}} \qquad \mathcal{L} = \leq_{\mathcal{L}} \cap \geq_{\mathcal{L}} \qquad \mathcal{J} = \leq_{\mathcal{J}} \cap \geq_{\mathcal{J}}$$
$$\mathcal{H} = \mathcal{R} \cap \mathcal{L} \qquad \mathcal{D} = \langle \mathcal{R} \cup \mathcal{L} \rangle$$

where the final expression denotes the equivalence relation generated by $\mathcal{R}$ and $\mathcal{L}$. Let $\mathbb{D}$ denote the set of equivalence classes of $\mathcal{D}$ in S, or $\mathcal{D}$ *classes*. For $D \in \mathbb{D}$, let $\mathbb{L}_D$ and $\mathbb{R}_D$ denote the sets of $\mathcal{L}$ and $\mathcal{R}$ classes in D respectively. The following property of Green's relations, along with its dual, constitutes a fundamental result known as Green's Lemma.

Lemma 1 (Green's Lemma [6])**.** *Suppose that $x \in S$ and $a \in S$ are such that $xa \mathrel{\mathcal{R}} x$. Then right multiplication by a gives an $\mathcal{R}$ class preserving bijection from the $\mathcal{L}$ class of x to the $\mathcal{L}$ class of xa.*

A semigroup S is said to be *group bound* if for each $x \in S$, there exists a positive integer n such that x^n lies in a subgroup of S. In particular, every finite semigroup is group bound. The following results are well known.

Theorem 2. *Suppose S is a group bound semigroup. Then*

(i) The relations $\mathcal{J}$ and $\mathcal{D}$ coincide.
(ii) If $x \mathrel{\mathcal{D}} xy$ then $x \mathrel{\mathcal{R}} xy$.
(iii) If $y \mathrel{\mathcal{D}} xy$ then $y \mathrel{\mathcal{L}} xy$.

Recall also that a semigroup S is *regular* if, for each $x \in S$, there exists $y \in S$ such that $xyx = x$. Equivalently S is regular if each $\mathcal{D}$ class contains an idempotent.

3 Twisted Semigroup Algebras

By analogy with twisted group algebras [15], we define a twisted semigroup algebra. The following definition is essentially that in [2], except that we give no special treatment to the zero of the semigroup (if it exists).

Definition 3. Suppose S is a semigroup and R is a commutative ring with 1. A *twisting* from S into R is a map

$$\alpha : S \times S \to R$$

which satisfies

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z) \tag{1}$$

for all $x, y, z \in S$. The *twisted semigroup algebra* of S over R, with twisting α, denoted by $R^\alpha[S]$, is the R-algebra with R-basis S and multiplication $\cdot$ defined by

$$x \cdot y = \alpha(x,y)(xy)$$

for $x, y \in S$, and extended by linearity. It follows easily from (1) that $R^\alpha[S]$ is associative.

For $T \subseteq S$, let $R^\alpha[T]$ denote the R-span of T in $R^\alpha[S]$, so that T forms an R-basis for $R^\alpha[T]$. It is clear that if T is a subsemigroup of S, then $R^\alpha[T]$ is a subalgebra, and moreover is isomorphic to the twisted semigroup algebra of T whose twisting is the restriction of α to T, thus justifying the notation.

4 Cellular Algebras

Cellular algebras were introduced in the famous paper of Graham and Lehrer [5]. Although the definition in [5] requires the algebra to be unital, it is easy to see that this does not affect the theory significantly.

Definition 4. Suppose that R is a commutative ring with identity. Recall that an anti-involution $*$ on an R-algebra A is an R-linear map from A to A such that

$$(a^*)^* = a \qquad \text{and} \qquad (ab)^* = b^*a^*$$

for a and $b \in A$. An associative R-algebra A is *cellular*, with *cell datum* $(\Lambda, M, C, *)$, if

(C1) Λ is a finite poset, and for each $\lambda \in \Lambda$ we have a finite indexing set $M(\lambda)$ and elements $C^\lambda_{st} \in A$ for $s, t \in M(\lambda)$. The elements

$$\{C^\lambda_{st} \mid \lambda \in \Lambda \text{ and } s, t \in M(\lambda)\}$$

form an R-basis of A.

(C2) The map $* : A \to A$ is an anti-involution, whose action on the above basis is given by

$$\left(C^{\lambda}_{st}\right)^{*} = C^{\lambda}_{ts}.$$

(C3) For any $\lambda \in \Lambda$, $s \in M(\lambda)$ and $a \in A$, there exist elements $r_a(s', s) \in R$ for $s' \in M(\lambda)$ such that, for each $t \in M(\lambda)$,

$$aC^{\lambda}_{st} \in \sum_{s' \in M(\lambda)} r_a(s', s) C^{\lambda}_{s't} + A(< \lambda)$$

where

$$A(< \lambda) = \mathrm{span}_R\{C^{\mu}_{s''t''} \mid \mu < \lambda \text{ and } s'', t'' \in M(\mu)\}.$$

5 The Main Theorem

In this section we prove a version of Theorem 15 of [3] for regular semigroups and for twisted semigroup algebras. As in [3], we will assume that the group algebras of the maximal subgroups of S are cellular, namely in Assumption 5. However, in [3] the anti-involutions on these algebras are woven together in the hope of creating an anti-involution on the semigroup algebra. In contrast, we start at the top by constructing an anti-involution $*$ on the semigroup algebra, and assuming that the anti-involution on each group algebra is a restriction of $*$. Therefore the assumptions we make will ensure that there is an anti-involution on the semigroup which induces an anti-involution on the twisted semigroup algebra, and which fixes certain maximal subgroups setwise.

We find it convenient to list the assumptions in the following discussion before stating the theorem. Firstly, we begin with the following objects.

Assumption 1. *Let S be a finite semigroup, $* : S \to S$ an anti-involution, R a commutative ring with identity, and α a twisting from S into R.*

We suppose that $*$ and α are compatible in the following sense.

Assumption 2. *Assume that*

$$\alpha(x, y) = \alpha(y^*, x^*)$$

for all $x, y \in S$.

This assumption implies that $*$ extends to an R-linear anti-involution on $R^{\alpha}[S]$, which we also denote by $*$. The next assumption ensures that $*$ fixes certain maximal subgroups, and also that S is regular.

Assumption 3. *Suppose that for each $\mathcal{D}$ class $D \in \mathbb{D}$, we have an idempotent $1_D \in D$ which is fixed by $*$.*

Let L_D denote the $\mathcal{L}$ class of 1_D, so L_D^* is the $\mathcal{R}$ class of 1_D. The $\mathcal{H}$ class $G_D = L_D \cap L_D^*$ of 1_D is a group. Moreover $*$ fixes G_D, and we denote its restriction to G_D by $*$. We will need a certain twisted group algebra over G_D to be cellular. However, for this to give information about the rest of the $\mathcal{D}$ class D, we need the scalar elements $\alpha(x, y)$ to be "sufficiently invertible". The following assumption, although very unnatural, gives us the generality we require. It essentially says that although α may not be invertible, when restricted to $L_D \times L_D^*$ it can be decomposed into a constant part and an invertible part. We use $G(R)$ to denote the group of units of R.

Assumption 4. *For each $\mathcal{D}$ class D, we assume the existence of a map*

$$\beta : L_D \times L_D^* \to G(R)$$

which satisfies the following analogues of (1) and Assumption 2:

$$\beta(x, y)\beta(xy, z) = \beta(x, yz)\beta(y, z), \tag{2}$$
$$\alpha(x, y)\beta(xy, z) = \alpha(x, yz)\beta(y, z), \text{ and} \tag{3}$$
$$\beta(x, y) = \beta(y^*, x^*) \tag{4}$$

whenever the relevant values of β are defined.

Before proceeding, we discuss the implications of Assumption 4. By (2), the restriction of β defines a twisting from G_D into R. Also as above, (4) implies that $*$ induces an anti-involution on $R^\beta[G_D]$, which we again denote by $*$. Now replacing x, y and z with z^*, y^* and x^* respectively in (3), and employing Assumption 2 and (4), we obtain

$$\beta(x, yz)\alpha(y, z) = \beta(x, y)\alpha(xy, z) \tag{5}$$

whenever the values of β are defined. As foreshadowed, the restriction of α to $L_D \times L_D^*$ can be obtained from β by multiplying by a constant. Indeed putting $x = y = 1_D$ in (3), we obtain

$$\alpha(1_D, 1_D)\beta(1_D, z) = \alpha(1_D, z)\beta(1_D, z)$$

for $z \in L_D^*$. Since $\beta(1_D, z)$ is invertible, this gives $\alpha(1_D, z) = \alpha(1_D, 1_D)$. Similarly putting $y = z = 1_D$ in (2) gives $\beta(x, 1_D) = \beta(1_D, 1_D)$ for $x \in L_D$. Finally for $x \in L_D$ and $z \in L_D^*$, putting $y = 1_D$ in (5) gives

$$\beta(x, z)\alpha(1_D, z) = \beta(x, 1_D)\alpha(x, z).$$

Thus $\beta(x,z)\alpha(1_D,1_D) = \beta(1_D,1_D)\alpha(x,z)$, so that

$$\alpha(x,z) = \alpha(D)\beta(x,z),$$

where $\alpha(D) = \alpha(1_D,1_D)\beta(1_D,1_D)^{-1}$. In particular, multiplication by $\alpha(D)$ gives a homomorphism $R^\alpha[G_D] \to R^\beta[G_D]$.

As foreshadowed, our final assumption is that certain twisted group algebras of the maximal subgroups are cellular.

Assumption 5. *Suppose that, for each $\mathcal{D}$ class D, the twisted group algebra $R^\beta[G_D]$ is cellular with cell datum*

$$(\Lambda_D, M_D, C, *).$$

Note we have assumed that the anti-involution in this cell datum is exactly $*$. Under these assumptions, we will show that the twisted semigroup algebra $R^\alpha[S]$ is cellular. To be more precise, we describe the cell datum below. Because S is finite, $\mathcal{D} = \mathcal{J}$ by (i) of Theorem 2, so we have a relation $\leq_{\mathcal{D}}$ on S. Define the poset

$$\Lambda = \{(D,\lambda) \mid D \in \mathbb{D} \text{ and } \lambda \in \Lambda_D\}$$

with partial order

$$(D_1,\lambda_1) \leq (D_2,\lambda_2) \text{ iff } D_1 <_{\mathcal{D}} D_2 \text{ or } D_1 = D_2 \text{ and } \lambda_1 \leq \lambda_2 \text{ in } \Lambda_{D_1}.$$

Now for $(D,\lambda) \in \Lambda$, let

$$M(D,\lambda) = \mathbb{L}_D \times M_D(\lambda).$$

Finally for each $L \in \mathbb{L}_D$, choose any $u_L \in L$ with $u_L \mathrel{\mathcal{R}} 1_D$. The basis elements that result from the cell datum of $R^\beta[G_D]$ can be written uniquely as

$$C^\lambda_{st} = \sum_{g \in G_D} c^\lambda_{st}(g) g$$

for some coefficients $c^\lambda_{st}(g) \in R$. Define

$$C^{(D,\lambda)}_{(L,s)(K,t)} = \sum_{g \in G_D} c^\lambda_{st}(g)\beta(u_L^*, g)\beta(u_L^* g, u_K)(u_L^* g u_K) \in R^\alpha[S]$$

for each $(D,\lambda) \in \Lambda_D$ and $(L,s),(K,t) \in M(D,\lambda)$.

Theorem 5. *Under Assumptions 1, 2, 3, 4 and 5, the algebra $R^\alpha[S]$ is cellular with the cell datum*

$$(\Lambda, M, C, *)$$

as given above.

As mentioned, Assumption 4 is very unnatural. However, we are primarily interested in two special cases. The first is the most natural, and applies when the twisting elements $\alpha(x, y)$ are invertible. In particular this includes the case of a semigroup algebra, in which the twisting is trivial.

Corollary 6. *Suppose Assumptions 1, 2 and 3 hold. Suppose also that for each $D \in \mathbb{D}$ and for each $x \mathcal{L} 1_D$ and $y \mathcal{R} 1_D$, the element $\alpha(x, y) \in R$ is invertible. As in Assumption 5, suppose that $R^\alpha[G_D]$ is cellular with cell datum*

$$(\Lambda_D, M_D, C, *).$$

Then the algebra $R^\alpha[S]$ is cellular with the cell datum

$$(\Lambda, M, C, *),$$

where Λ, M and $*$ are as given above. The basis elements now take the more elegant form

$$C^{(D,\lambda)}_{(L,s)(K,t)} = u_L^* \cdot C^\lambda_{st} \cdot u_K.$$

This follows from Theorem 5 by setting β to be the relevant restriction of α for each D class. The second special case will aid our investigation of the Brauer, Temperley-Lieb and partition algebras.

Corollary 7. *Suppose Assumptions 1, 2 and 3 hold. Suppose also that we have $\alpha(x, y) = \alpha(x, z)$ whenever $y \mathcal{R} z$. Suppose that the group algebra $R[G_D]$ is cellular with cell datum*

$$(\Lambda_D, M_D, C, *).$$

Then the algebra $R^\alpha[S]$ is cellular with the cell datum

$$(\Lambda, M, C, *),$$

where Λ, M and $*$ are as given above. The basis elements now take the form

$$C^{(D,\lambda)}_{(L,s)(K,t)} = \sum_{g \in G_D} c^\lambda_{st}(g)(u_L^* g u_K).$$

This follows from Theorem 5 be setting $\beta(x, y) = 1$. To verify (3) of Assumption 4 in this case, suppose $\beta(xy, z)$ and $\beta(y, z)$ are defined, so that $y \in L_D$ and $z \in L_D^*$ for some $D \in \mathbb{D}$. Then $1_D z = z$, so Green's Lemma shows that right multiplication by z is an $\mathcal{R}$ class preserving map on L_D. In particular $yz \mathcal{R} y$, so $\alpha(x, y) = \alpha(x, yz)$ as required.

The proof of Theorem 5 contains many notationally unpleasant calculations related to associativity. To partially alleviate this, we introduce a partial product on $R^\alpha[S]$. For each $D \in \mathbb{D}$, define

$$\circ : R^\alpha[L_D] \times R^\alpha[L_D^*] \to R^\alpha[S]$$

by setting $x \circ y = \beta(x, y)(xy)$ for $x \in L_D$ and $y \in L_D^*$, and extending by R-linearity. It will often be necessary to check that the arguments of $\circ$ lie in $R^\alpha[L_D]$ and $R^\alpha[L_D^*]$ respectively, for the appropriate D; we generally leave this to the reader. It should be noted that in the special case of Corollary 6, this product coincides with $\cdot$, so the associativity of $\cdot$ makes many of the tedious calculations trivial; thus a direct proof of this case is much more natural, and still contains the essential ideas.

Note that $R^\beta[G_D]$ is equal to $R^\alpha[G_D]$ as an R-module, and the product on $R^\beta[G_D]$ is just the restriction of $\circ$. Also the above definition of $C^{(D,\lambda)}_{(L,s)(K,t)}$ now becomes

$$C^{(D,\lambda)}_{(L,s)(K,t)} = (u_L^* \circ C^\lambda_{st}) \circ u_K$$

for $(D, \lambda) \in \Lambda_D$ and $(L, s), (K, t) \in M(D, \lambda)$. Applying linearity to equations (2), (3), (4) and (5) respectively, we obtain:

$$(a \circ b) \circ c = a \circ (b \circ c), \tag{6}$$

$$(a \cdot b) \circ c = a \cdot (b \circ c) \text{ if } (\operatorname{supp} a)(\operatorname{supp} b) \subseteq L_D, \tag{7}$$

$$(a \circ b)^* = b^* \circ a^*, \text{ and} \tag{8}$$

$$(a \circ b) \cdot c = a \circ (b \cdot c) \text{ if } (\operatorname{supp} b)(\operatorname{supp} c) \subseteq L_D^* \tag{9}$$

for any a, b and $c \in R^\alpha[S]$, whenever the relevant values of $\circ$ are defined. Here $\operatorname{supp} a$ is the set of elements of S which appear with nonzero coefficient in a. We now give a proof Theorem 5, which for clarity we separate into three lemmas corresponding to properties (C1), (C2) and (C3) of Definition 4.

Lemma 8. *The elements*

$$\left\{ C^{(D,\lambda)}_{(L,s)(K,t)} \;\middle|\; (D, \lambda) \in \Lambda \text{ and } (L, s), (K, t) \in M(D, \lambda) \right\}$$

form an R-basis for $R^\alpha[S]$.

Proof. Consider a $\mathcal{D}$ class $D \in \mathbb{D}$. Now $*$ preserves $\mathcal{D}$ and $1_D^* = 1_D$, so $*$ maps D onto D. Since $*$ is an anti-involution, it therefore maps the $\mathcal{L}$ classes in D bijectively onto the $\mathcal{R}$ classes in D. That is, each $\mathcal{R}$ class in D is uniquely expressible as L^* for some $L \in \mathbb{L}_D$. Thus each $\mathcal{H}$-class in D is uniquely expressible as $L^* \cap K$ for some $L, K \in \mathbb{L}_D$.

For each $L \in \mathbb{L}_D$, we have $u_L \,\mathcal{R}\, 1_D$ by choice of u_L. Since 1_D is idempotent, this implies that $1_D u_L = u_L$. By Green's Lemma, right multiplication by u_L then gives an $\mathcal{R}$ class preserving bijection from the $\mathcal{L}$ class of 1_D to the $\mathcal{L}$ class of u_L, which is L. Applying $*$ we have $u_L^* 1_D = u_L^*$, so left multiplication by u_L^* gives an $\mathcal{L}$ class preserving bijection from the $\mathcal{R}$ class of 1_D to the $\mathcal{R}$ class of u_L^*, namely L^*. We therefore have two bijections

$$G_D \to L^* \cap L_D \to L^* \cap K$$

given respectively by $g \mapsto u_L^* g$ and $x \mapsto x u_K$. Thus we have R-module homomorphisms

$$R^\beta[G_D] = R^\alpha[G_D] \to R^\alpha[L^* \cap L_D] \to R^\alpha[L^* \cap K]$$

given respectively by $a \mapsto u_L^* \circ a$ and $a \mapsto a \circ u_K$. On the natural bases these homomorphisms are given by

$$\begin{aligned} g &\mapsto \beta(u_L^*, g)(u_L^* g) &\quad& \text{for } g \in G_D, \text{ and} \\ x &\mapsto \beta(x, u_K)(x u_K) && \text{for } x \in L^* \cap L_D. \end{aligned}$$

Because the elements $\beta(x, y)$ are invertible, and the above maps between the natural bases are bijections, these homomorphisms are R-module isomorphisms. Now the elements

$$\left\{ C_{st}^\lambda \;\middle|\; \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\}$$

form an R-basis for $R^\beta[G_D]$, so applying the above isomorphisms, the elements

$$\begin{aligned} &\left\{ \left(u_L^* \circ C_{st}^\lambda \right) \circ u_K \;\middle|\; \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} \\ &\qquad\qquad = \left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \;\middle|\; \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} \end{aligned}$$

form an R-basis for $R^\alpha[L^* \cap K]$. Now D is a disjoint union of its $\mathcal{H}$ classes

$$D = \coprod_{L, K \in \mathbb{L}_D} L^* \cap K,$$

and S is in turn a disjoint union of its $\mathcal{D}$ classes

$$S = \coprod_{D \in \mathbb{D}} D.$$

Thus

$$R^\alpha[D] = \bigoplus_{L, K \in \mathbb{L}_D} R^\alpha[L^* \cap K]$$

and

$$R^\alpha[S] = \bigoplus_{D \in \mathbb{D}} R^\alpha[D],$$

so that

$$\left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \;\middle|\; \lambda \in \Lambda_D \text{ and } (L, s), (K, t) \in M(D, \lambda) \right\} \tag{10}$$

form an R-basis for $R^\alpha[D]$, and

$$\left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \;\middle|\; (D, \lambda) \in \Lambda \text{ and } (L, s), (K, t) \in M(D, \lambda) \right\}$$

form an R-basis for $R^\alpha[S]$. □

This verifies property (C1) in Definition 4. We next prove property (C2). We already know that $*$ is an R-linear anti-involution of $R^\alpha[S]$, so we need only check the following.

Lemma 9. *The action of $*$ on the basis elements $C^{(D,\lambda)}_{(L,s)(K,t)}$ is given by*

$$\left(C^{(D,\lambda)}_{(L,s)(K,t)}\right)^* = C^{(D,\lambda)}_{(K,t)(L,s)}.$$

Proof. By Assumption 5, we have $\left(C^\lambda_{st}\right)^* = C^\lambda_{ts}$. Thus

$$\begin{aligned}
\left(C^{(D,\lambda)}_{(L,s)(K,t)}\right)^* &= \left(\left(u_L^* \circ C^\lambda_{st}\right) \circ u_K\right)^* \\
&= u_K^* \circ \left(\left(C^\lambda_{st}\right)^* \circ u_L\right) \text{ using (8) twice} \\
&= \left(u_K^* \circ C^\lambda_{ts}\right) \circ u_L \text{ by (6)} \\
&= C^{(D,\lambda)}_{(K,t)(L,s)},
\end{aligned}$$

as required. □

Suppose $D, D' \in \mathbb{D}$ satisfy $D' <_{\mathcal{D}} D$. Pick any $\lambda \in \Lambda_D$. By (10), we have

$$\begin{aligned}
R^\alpha[D'] &= \operatorname{span}_R\left\{C^{(D',\lambda')}_{(L',s')(K',t')}\right\}. \\
&\subseteq \operatorname{span}_R\left\{C^{(D'',\lambda'')}_{(L'',s'')(K'',t'')} \;\middle|\; D'' <_{\mathcal{D}} D\right\}. \\
&\subseteq R^\alpha[S](< (D,\lambda)),
\end{aligned}$$

where $R^\alpha[S](< (D,\lambda))$ is as defined in Definition 4. Thus

$$\bigoplus_{D' <_{\mathcal{D}} D} R^\alpha[D'] \subseteq R^\alpha[S](< (D,\lambda)). \tag{11}$$

We now prove (C3).

Lemma 10. *Given $(D,\lambda) \in \Lambda$ and $(L,s) \in M(D,\lambda)$, and for an element $a \in R^\alpha[S]$, there exist elements $r_a((L',s'),(L,s)) \in R$ for $(L',s') \in M(D,\lambda)$ such that*

$$a \cdot C^{(D,\lambda)}_{(L,s)(K,t)} \in \sum_{(L',s') \in M(D,\lambda)} r_a((L',s'),(L,s)) C^{(D,\lambda)}_{(L',s')(K,t)} + R^\alpha[S](< (D,\lambda))$$

for each $(K,t) \in M(D,\lambda)$.

Proof. Because S spans $R^\alpha[S]$ as an R-module, it suffices to take $a \in S$. Because $u_L^* \in D$, clearly $au_L^* \leq_{\mathcal{D}} D$. First suppose that $au_L^* <_{\mathcal{D}} D$. Then $au_L^* g u_K <_{\mathcal{D}} D$ for all $g \in G_D$ and $K \in \mathcal{L}_D$, so (11) gives

$$\alpha(a, u_L^* g u_K) c_{st}^\lambda(g) \beta(u_L^*, g) \beta(u_L^* g, u_K)(au_L^* g u_K) \in R^\alpha[S](< (D, \lambda))$$

for $t \in M_D(\lambda)$. Summing over $g \in G_D$ gives $a \cdot C_{(L,s)(K,t)}^{(D,\lambda)} \in R^\alpha[S](< (D, \lambda))$. It therefore suffices to take $r_a((L', s'), (L, s)) = 0$ for all $(L', s') \in M(D, \lambda)$ in this case.

The other case is when $au_L^* \in D$. It follows from (iii) of Theorem 2 that $au_L^* \mathrel{\mathcal{L}} u_L^*$, so that $au_L^* \in L_D$. Thus if $L_1^* \in \mathbb{R}_D$ is the $\mathcal{R}$ class of au_L^*, then $au_L^* \mathrel{\mathcal{H}} u_{L_1}^*$. As in the proof of Lemma 8 above, it follows from Green's Lemma that $au_L^* = u_{L_1}^* h$ for some $h \in G_D$. By Assumption 5, there exist ring elements $r_h(s', s) \in R$ for $s' \in M_D(\lambda)$ such that

$$\begin{aligned} h \circ C_{st}^\lambda - \sum_{s' \in M_D(\lambda)} r_h(s', s) C_{s't}^\lambda &\in R^\beta[G_D](< \lambda) \\ &= \operatorname{span}_R \left\{ C_{uv}^\mu \,|\, \mu < \lambda \text{ and } u, v \in M_D(\mu) \right\}. \end{aligned}$$

Applying $u_{L_1}^* \circ$ on the left and $\circ\, u_K$ on the right, we obtain

$$\begin{aligned} &\left(u_{L_1}^* \circ \left(h \circ C_{st}^\lambda\right)\right) \circ u_K - \sum_{s' \in M_D(\lambda)} r_h(s', s) C_{(L_1, s')(K,t)}^{(D,\lambda)} \\ &\quad \in \operatorname{span}_R \left\{ C_{(L_1,u)(K,v)}^{(D,\mu)} \,\middle|\, \mu < \lambda \text{ and } u, v \in M_D(\mu) \right\} \\ &\quad \subseteq R^\alpha[S](< (D, \lambda)). \end{aligned}$$

We can also calculate

$$\begin{aligned} u_{L_1}^* \circ \left(h \circ C_{st}^\lambda\right) &= \left(u_{L_1}^* \circ h\right) \circ C_{st}^\lambda \text{ by (6)} \\ &= \beta(u_{L_1}^*, h) \left(u_{L_1}^* h\right) \circ C_{st}^\lambda \\ &= \beta(u_{L_1}^*, h) \left(au_L^*\right) \circ C_{st}^\lambda. \end{aligned}$$

Combining these, we obtain

$$
\begin{aligned}
a \cdot C^{(D,\lambda)}_{(L,s)(K,t)} &= a \cdot \left(u^*_L \circ \left(C^\lambda_{st} \circ u_K\right)\right) \\
&= (a \cdot u^*_L) \circ \left(C^\lambda_{st} \circ u_K\right) \text{ by (7)} \\
&= \alpha(a, u^*_L)\, (au^*_L) \circ \left(C^\lambda_{st} \circ u_K\right) \\
&= \alpha(a, u^*_L) \left((au^*_L) \circ C^\lambda_{st}\right) \circ u_K \text{ by (6)} \\
&= \alpha(a, u^*_L)\beta(u^*_{L_1}, h)^{-1} \left(u^*_{L_1} \circ \left(h \circ C^\lambda_{st}\right)\right) \circ u_K \\
&\in \alpha(a, u^*_L)\beta(u^*_{L_1}, h)^{-1} \sum_{s' \in M_D(\lambda)} r_h(s', s) C^{(D,\lambda)}_{(L_1,s')(K,t)} \\
&\qquad\qquad + R^\alpha[S](< (D, \lambda)).
\end{aligned}
$$

It therefore suffices to take

$$
r_a((L', s'), (L, s)) = \begin{cases} \alpha(a, u^*_L)\beta(u^*_{L_1}, h)^{-1} r_h(s', s) & \text{if } L' = L_1 \\ 0 & \text{if } L' \neq L_1. \end{cases}
$$

□

6 Linear Representations of Regular Semigroups

Section 2 of [5] describes how to construct *cell representations* of a cellular algebra A from its cell datum $(\Lambda, M, C, *)$, and defines bilinear forms ϕ^λ associated with these representations. For convenience we reproduce the definitions here. For each $\lambda \in \Lambda$, the cell representation $W(\lambda)$ corresponding to λ is the left A-module with R-basis $\{C_s \mid s \in M(\lambda)\}$ and A-action

$$
aC_s = \sum_{s' \in M(\lambda)} r_a(s', s) C_{s'}
$$

for $a \in A$ and $s \in M(\lambda)$. We use

$$
\rho^\lambda : A \to \mathrm{Mat}_{M(\lambda)}(R)
$$

to denote the corresponding representation relative to the natural basis. That is,

$$
\rho^\lambda(a)_{st} = r_a(s, t)
$$

for $a \in A$ and $s, t \in M(\lambda)$. For each $a \in A$, the bilinear form ϕ^λ_a on $W(\lambda)$ is defined on the basis elements so that $\phi^\lambda_a(C_s, C_t)$ is the unique element of R satisfying

$$
C^\lambda_{s's} a C^\lambda_{tt'} \in \phi^\lambda_a(C_s, C_t) C^\lambda_{s't'} + A(< \lambda) \tag{12}
$$

for all $s', t' \in M(\lambda)$. This is extended to be R-bilinear. We are most interested in the bilinear form

$$
\phi^\lambda = \phi^\lambda_1.
$$

We use Φ^λ to denote the matrix representation of ϕ^λ relative to the natural basis. That is, $\Phi^\lambda \in \mathrm{Mat}_{M(\lambda)}(R)$ is defined by

$$\Phi^\lambda_{st} = \phi^\lambda(C_s, C_t)$$

for $s, t \in M(\lambda)$. In fact ϕ^λ_a can be related to ϕ^λ and ρ^λ using (C3) of Definition 4. More precisely,

$$\phi^\lambda_a(C_s, C_t) = \sum_{t' \in M(\lambda)} \Phi^\lambda_{st'} \rho^\lambda(a)_{t't} \tag{13}$$

for $a \in A$ and $s, t \in M(\lambda)$. The importance of ϕ^λ is demonstrated by the following theorem.

Theorem 11 ([5] Theorem 3.8)**.** *In the above notation, if R is a field then the following are equivalent.*

(i) The algebra A is semisimple.
(ii) The nonzero cell representations $W(\lambda)$ are irreducible and pairwise inequivalent.
(iii) The form ϕ^λ is nondegenerate (ie $\det \Phi^\lambda \neq 0$) for each $\lambda \in \Lambda$.

Theorem 5 allows us to obtain cell representations of $R^\alpha[S]$ from the cell representations of the twisted group algebras $R^\beta[G_D]$. In fact this is a special case of the following general result, the proof of which is a consequence of Theorem 2 and Green's Lemma, and is omitted (see also Theorem 2.3 of [13]).

Proposition 12. *Suppose that S is any group bound semigroup and that D is a regular $\mathcal{D}$ class in S with maximal subgroup G. Suppose α is a twisting from S into R, and*

$$\beta : L_D \times K_D \to G(R)$$

is a map satisfying (2) and (3), where L_D is the $\mathcal{L}$ class of G and K_D is the $\mathcal{R}$ class of G. Suppose that M is a left $R^\beta[G]$-module. For each $K \in \mathbb{R}_D$, pick an element $v_K \in K$ in the same $\mathcal{L}$ class as G, and let

$$M_K = \{m_K \mid m \in M\}$$

be a set in bijection with M. Then

$$W = \bigoplus_{K \in \mathbb{R}_D} M_K$$

is a left $R^\alpha[S]$-module under the action which is defined on S by

$$s \cdot m_K = \begin{cases} 0 & \text{if } sv_K <_{\mathcal{D}} D \\ \alpha(s, v_K)\beta(v_{K'}, g)^{-1}(gm)_{K'} & \text{if } sv_K = v_{K'}g \text{ where } g \in G \end{cases}$$

for $m \in M$, $K \in \mathbb{R}_D$ and $s \in S$, and which is extended to $R^\alpha[S]$ by R-linearity.

Now suppose that the assumptions of Theorem 5 hold. By analogy with Section 4 of [3], we determine the bilinear forms associated with the cell representations of $R^\alpha[S]$ in terms of the cell representations of $R^\beta[G_D]$. For any $L, K \in \mathbb{L}_D$ it follows from (ii) and (iii) of Theorem 2 that the element $u_L u_K^*$ is either in G_D or in a lower $\mathcal{D}$ class than D. We can therefore define the matrix $P_D^\alpha \in \mathrm{Mat}_{\mathbb{L}_D}(R^\beta[G_D])$ by

$$(P_D^\alpha)_{LK} = \begin{cases} 0 & \text{if } u_L u_K^* <_{\mathcal{D}} D \\ \alpha(u_L, u_K^*) u_L u_K^* & \text{if } u_L u_K^* \in G_D. \end{cases}$$

Call P_D^α the *twisted sandwich matrix* of D. Of course when α is trivial, this reduces to the usual sandwich matrix, on identifying $G_D \cup \{0\}$ with a subset of $R[G_D]$. We can now state the analogue of Lemma 16 of [3].

Lemma 13. *Let $(D, \lambda) \in \Lambda$ and $(L, s), (K, t) \in M(D, \lambda)$. Then*

$$\phi^{(D,\lambda)}\left(C_{(L,s)}, C_{(K,t)}\right) = \phi^\lambda_{(P_D^\alpha)_{LK}}\left(C_s, C_t\right).$$

Proof. Suppose first that $u_L u_K^* <_{\mathcal{D}} D$, so $(P_D^\alpha)_{LK} = 0$. Then (11) gives

$$u_K^* g u_L u_K^* h u_L \in R^\alpha[S](< (D, \lambda))$$

for g, $h \in G_D$. Multiplying by

$$\alpha(u_K^* g u_L, u_K^* h u_L) c^\lambda_{ts}(g) \beta(u_K^*, g) \beta(u_K^* g, u_L) c^\lambda_{ts}(h) \beta(u_K^*, h) \beta(u_K^* h, u_L)$$

and summing over g and h, we obtain

$$C^{(D,\lambda)}_{(K,t)(L,s)} \cdot C^{(D,\lambda)}_{(K,t)(L,s)} \in R^\alpha[S](< (D, \lambda)).$$

Thus

$$\phi^{(D,\lambda)}\left(C_{(L,s)}, C_{(K,t)}\right) = 0 = \phi^\lambda_{(P_D^\alpha)_{LK}}\left(C_s, C_t\right)$$

in this case. The other case is when $u_L u_K^* \in G_D$. Then $(P_D^\alpha)_{LK} = u_L \cdot u_K^*$, so (7) gives

$$(P_D^\alpha)_{LK} \circ C^\lambda_{ts} = (u_L \cdot u_K^*) \circ C^\lambda_{ts} = u_L \cdot \left(u_K^* \circ C^\lambda_{ts}\right).$$

Now $u_L u_K^* g \in G_D \subseteq L_D^*$ for all $g \in G_D$. Thus applying $\circ\, u_L$ on the right,

$$\begin{aligned} \left((P_D^\alpha)_{LK} \circ C^\lambda_{ts}\right) \circ u_L &= \left(u_L \cdot \left(u_K^* \circ C^\lambda_{ts}\right)\right) \circ u_L \\ &= u_L \cdot \left(\left(u_K^* \circ C^\lambda_{ts}\right) \circ u_L\right) \text{ by (7)} \\ &= u_L \cdot C^{(D,\lambda)}_{(K,t)(L,s)}. \end{aligned}$$

Because $u_L u_K^* g \in G_D$, it follows that $u_L u_K^* g u_L \in L_D^*$ as in the proof of Lemma 8. Thus applying $\left(u_K^* \circ C^\lambda_{ts}\right) \circ$ on the left gives

$$
\begin{aligned}
\left(u_K^* \circ C_{ts}^\lambda\right) &\circ \left(\left((P_D^\alpha)_{LK} \circ C_{ts}^\lambda\right) \circ u_L\right) \qquad (14)\\
&= \left(u_K^* \circ C_{ts}^\lambda\right) \circ \left(u_L \cdot C_{(K,t)(L,s)}^{(D,\lambda)}\right)\\
&= \left(\left(u_K^* \circ C_{ts}^\lambda\right) \circ u_L\right) \cdot C_{(K,t)(L,s)}^{(D,\lambda)} \text{ by (9)}\\
&= C_{(K,t)(L,s)}^{(D,\lambda)} \cdot C_{(K,t)(L,s)}^{(D,\lambda)}.
\end{aligned}
$$

Now by definition of ϕ^λ, we have

$$
C_{ts}^\lambda \circ (P_D^\alpha)_{LK} \circ C_{ts}^\lambda \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{ts}^\lambda + R^\beta[G_D](< \lambda).
$$

As in the proof of Lemma 10, applying $u_K^* \circ$ on the left and $\circ\, u_L$ gives

$$
\begin{aligned}
\left(u_K^* \circ \left(C_{ts}^\lambda \circ (P_D^\alpha)_{LK} \circ C_{ts}^\lambda\right)\right) \circ u_L \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{(K,t)(L,s)}^{(D,\lambda)}\\
+ R^\alpha[S](< (D,\lambda)).
\end{aligned}
$$

By applying (6) repeatedly, the left hand side is exactly (14). Therefore

$$
C_{(K,t)(L,s)}^{(D,\lambda)} \cdot C_{(K,t)(L,s)}^{(D,\lambda)} \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{(K,t)(L,s)}^{(D,\lambda)} + R^\alpha[S](< (D,\lambda)),
$$

whence the result. □

For each $\lambda \in \Lambda_D$, the representation

$$
\rho^\lambda : R^\beta[G_D] \to \mathrm{Mat}_{M_D(\lambda)}(R)
$$

naturally induces a homomorphism

$$
\mathrm{Mat}_{\mathbb{L}_D}(R^\beta[G_D]) \to \mathrm{Mat}_{\mathbb{L}_D}\left(\mathrm{Mat}_{M_D(\lambda)}(R)\right) \cong \mathrm{Mat}_{M(D,\lambda)}(R),
$$

which we also denote by ρ^λ.

Corollary 14. *The matrix representation of $\phi^{(D,\lambda)}$ is given by*

$$
\Phi^{(D,\lambda)} = \Phi'^\lambda \rho^\lambda(P_D^\alpha),
$$

where Φ'^λ is the block diagonal matrix

$$
\Phi'^\lambda = \begin{pmatrix} \Phi^\lambda & 0 & 0 & \cdots & 0 \\ 0 & \Phi^\lambda & 0 & \cdots & 0 \\ 0 & 0 & \Phi^\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi^\lambda \end{pmatrix} \in \mathrm{Mat}_{\mathbb{L}_D}\left(\mathrm{Mat}_{M_D(\lambda)}(R)\right) \cong \mathrm{Mat}_{M(D,\lambda)}(R).
$$

Thus

$$\det \Phi^{(D,\lambda)} = \left(\det \Phi^{\lambda}\right)^{|\mathbb{L}_D|} \det \rho^{\lambda}(P^{\alpha}_{D}).$$

Proof. Using (13), the previous Lemma gives

$$\begin{aligned}
\Phi^{(D,\lambda)}_{(L,s)(K,t)} &= \phi^{(D,\lambda)}\left(C_{(L,s)}, C_{(K,t)}\right)\\
&= \phi^{\lambda}_{(P^{\alpha}_{D})_{LK}}(C_s, C_t)\\
&= \sum_{t' \in M_D(\lambda)} \Phi^{\lambda}_{st'} \rho^{\lambda}\left((P^{\alpha}_{D})_{LK}\right)_{t't}\\
&= \sum_{t' \in M_D(\lambda)} \Phi^{\lambda}_{st'} \rho^{\lambda}\left(P^{\alpha}_{D}\right)_{(L,t')(K,t)}\\
&= \sum_{(L',t') \in M(D,\lambda)} \Phi^{\lambda}_{st'} \delta_{LL'} \rho^{\lambda}\left(P^{\alpha}_{D}\right)_{(L',t')(K,t)}\\
&= \sum_{(L',t') \in M(D,\lambda)} \Phi'^{\lambda}_{(L,s)(L',t')} \rho^{\lambda}(P^{\alpha}_{D})_{(L',t')(K,t)}\\
&= \left(\Phi'^{\lambda} \rho^{\lambda}(P^{\alpha}_{D})\right)_{(L,s)(K,t)}.
\end{aligned}$$

Hence

$$\Phi^{(D,\lambda)} = \Phi'^{\lambda} \rho^{\lambda}(P^{\alpha}_{D})$$

as required. Taking the determinant, it is then clear that

$$\det \Phi^{(D,\lambda)} = \det \Phi'^{\lambda} \det \rho^{\lambda}(P^{\alpha}_{D}) = \left(\det \Phi^{\lambda}\right)^{|\mathbb{L}_D|} \det \rho^{\lambda}(P^{\alpha}_{D}).$$

This completes the proof of Corollary 14. $\square$

The utility of cellular machinery will be illustrated by providing an alternative proof of a special case (Theorem 16 below) of the following difficult theorem.

Theorem 15. *Suppose that S is a finite regular semigroup, and suppose α is a twisting from S into some field R such that $\alpha(x, y) \neq 0$ for each $x, y \in S$. Consider a $\mathcal{D}$ class D in S, and choose any idempotent $1_D \in D$. The $\mathcal{H}$ class G_D of 1_D is a group. For each $L \in \mathbb{L}_D$, pick an element $u_L \in L$ with $u_L \mathcal{R} 1_D$. Similarly for $K \in \mathbb{R}_D$, pick $v_K \in K$ with $v_K \mathcal{L} 1_D$. The twisted sandwich matrix P^{α}_{D} is the $\mathbb{L}_D \times \mathbb{R}_D$ matrix with entries in $R^{\alpha}[G_D]$ given by*

$$(P^{\alpha}_{D})_{LK} = \begin{cases} 0 & \text{if } u_L v_K <_{\mathcal{D}} D \\ \alpha(u_L, v_K) u_L v_K & \text{if } u_L v_K \in G_D. \end{cases}$$

Then $R^{\alpha}[S]$ is semisimple exactly when the following two conditions hold for each $\mathcal{D}$ class D.

(i) $R^{\alpha}[G_D]$ is semisimple.

(ii) P_D^α *is square and invertible.*

This result is exactly analogous to the well known non-twisted version [14]. Indeed it is easy to check that if $S_1 \subseteq S_2$ are ideals of S such that $S_2 \setminus S_1$ is a single $\mathcal{D}$ class D, then the quotient

$$R^\alpha[S_2]/R^\alpha[S_1] \cong R_0^\alpha[S_2/S_1]$$

is a Munn ring over the ring $R^\alpha[G_D]$, with sandwich matrix P_D^α; here the notation $R_0^\alpha[S_2/S_1]$ denotes the contracted twisted semigroup algebra, defined analogously to a contracted semigroup algebra. The above theorem then follows from Theorem 4.7 of [14] (see also [16]).

If the assumptions of Corollary 6 hold, the resulting cellular structure is sufficient by itself to quickly obtain the above theorem from general cellular algebra results, as we see below. Note that setting $v_{L^*} = u_L^*$, the definition of P_D^α given before Lemma 13 agrees with that in the above theorem.

Theorem 16. *Suppose that the conditions of Corollary 6 hold, and that R is a field. Then $R^\alpha[S]$ is semisimple exactly when*

(i) $R^\alpha[G_D]$ *is semisimple and*
(ii) P_D^α *is invertible,*

for each $D \in \mathbb{D}$, where P_D^α is as defined immediately before Lemma 13.

Proof. Suppose that the two conditions hold, and consider any $(D, \lambda) \in \Lambda$. Since P_D^α is invertible, certainly $\rho^\lambda(P_D^\alpha)$ is invertible. Thus $\det \rho^\lambda(P_D^\alpha) \neq 0$. Also because $R^\alpha[G_D]$ is semisimple, by Theorem 11 we have $\det \Phi^\lambda \neq 0$. Hence Corollary 14 gives

$$\det \Phi^{(D,\lambda)} \neq 0.$$

As this holds for each $(D, \lambda) \in \Lambda$, the algebra $R^\alpha[S]$ is semisimple by Theorem 11.

Conversely suppose that $R^\alpha[S]$ is semisimple, so that $\det \Phi^{(D,\lambda)} \neq 0$ for each $(D, \lambda) \in \Lambda$ by Theorem 11. By Corollary 14, we then have

$$\det \Phi^\lambda \neq 0 \qquad \text{and} \qquad \det \rho^\lambda(P_D^\alpha) \neq 0.$$

Now the former holds for all $\lambda \in \Lambda_D$. Thus applying Theorem 11, statement (i) implies that $R^\alpha[G_D]$ is semisimple, and moreover statement (iii) implies that the map

$$\bigoplus_{\lambda \in \Lambda_D} \rho^\lambda : R^\alpha[G_D] \to \bigoplus_{\lambda \in \Lambda_D} \mathrm{Mat}_{M(\lambda)}(R)$$

is an isomorphism. Because $\det \rho^\lambda(P_D^\alpha) \neq 0$, the matrix $\rho^\lambda(P_D^\alpha)$ is invertible

for each $\lambda \in \Lambda_D$. Thus

$$\bigoplus_{\lambda\in\Lambda_D} \rho^\lambda(P_D^\alpha) \in \bigoplus_{\lambda\in\Lambda_D} \mathrm{Mat}_{M(D,\lambda)}(R)$$

is invertible. The above isomorphism then implies that the matrix P_D^α is invertible. Thus both conditions hold, verifying the reverse direction and completing the proof of Theorem 16. □

7 The Partition Algebra

Fix an integer $n \geq 1$. For convenience, we denote

$$\begin{aligned} I &= \{1, 2, 3, \ldots, n\}, \\ I' &= \{1', 2', 3', \ldots, n'\}, \\ I'' &= \{1'', 2'', 3'', \ldots, n''\}. \end{aligned}$$

Let A_n denote the set of equivalence relations on the set $I \cup I'$. For $x \in A_n$, let $\tilde{x}$ denote the set of equivalence classes of x. We define a binary operation on A_n as follows. Consider two elements $x, y \in A_n$. Let y' denote the equivalence relation on the set $I' \cup I''$ which is obtained from y by appending a $'$ to each number. Let $\langle x \cup y' \rangle$ denote the equivalence relation on the set $I \cup I' \cup I''$ which is generated by x and y'. Let $m(x, y)$ denote the number of equivalence classes of $\langle x \cup y' \rangle$ which contain only single dashed elements, that is which are contained in I'. Remove all the single dashed elements from $\langle x \cup y' \rangle$ and replace the double dashes with single dashes to obtain xy. That is, xy is obtained from

$$\{(i, j) \in \langle x \cup y' \rangle \mid i, j \in I \cup I''\}$$

by replacing i'' with i'. For example, consider the elements $x, y \in A_7$ whose equivalence classes are

$$\begin{aligned} \tilde{x} &= \{\{1, 3, 4', 6'\}, \{2\}, \{4, 5, 6\}, \{7\}, \{1'\}, \{2', 3'\}, \{5', 7'\}\}, \\ \tilde{y} &= \{\{1\}, \{2, 4\}, \{3, 3', 4', 6'\}, \{5, 7\}, \{6, 5', 7'\}, \{1'\}, \{2'\}\}. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{y'} &= \{\{1'\}, \{2', 4'\}, \{3', 3'', 4'', 6''\}, \{5', 7'\}, \{6', 5'', 7''\}, \{1''\}, \{2''\}\}, \\ \widetilde{\langle x \cup y' \rangle} &= \{\{1, 3, 2', 3', 4', 6', 3'', 4'', 5'', 6'', 7''\}, \{2\}, \{4, 5, 6\}, \\ &\qquad \{7\}, \{1'\}, \{5', 7'\}, \{1''\}, \{2''\}\}, \\ \widetilde{xy} &= \{\{1, 3, 3', 4', 5', 6', 7'\}, \{2\}, \{4, 5, 6\}, \{7\}, \{1'\}, \{2'\}\}. \end{aligned}$$

Also $m(x, y) = 2$ since $\langle x \cup y' \rangle$ has two equivalence classes contained in I', namely $\{1'\}$ and $\{5', 7'\}$. This operation has a natural diagrammatic interpretation described in [11]. It is associative, and we have the relation

$$m(x, y) + m(xy, z) = m(x, yz) + m(y, z)$$

for any $x, y, z \in A_n$. The latter implies that for any δ in a commutative ring R, we can define a twisting from A_n into R by

$$\alpha(x, y) = \delta^{m(x,y)}.$$

The resulting twisted semigroup algebra $R^\alpha[A_n]$ is called the *partition algebra* [11]. This algebra was shown to be cellular by Xi in [20]. We reproduce this result here with the aid of Theorem 5.

We first note that A_n has a natural anti-involution $*$ which swaps i and i', for each $i \in I$. It is easy to see that α and $*$ satisfy Assumption 2. Green's relations in A_n are described by the following theorem, the proof of which is straightforward and omitted.

Theorem 17. *For $x \in A_n$, define the functions*

$$\begin{aligned}
d(x) &= \#\{J \in \tilde{x} \mid J \cap I \neq \emptyset \neq J \cap I'\},\\
r(x) &= (\{J \in \tilde{x} \mid J \subseteq I\}, \{J \cap I \mid J \in \tilde{x} \textit{ and } J \cap I \neq \emptyset \neq J \cap I'\}),\\
l(x) &= (\{J \in \tilde{x} \mid J \subseteq I'\}, \{J \cap I' \mid J \in \tilde{x} \textit{ and } J \cap I \neq \emptyset \neq J \cap I'\}).
\end{aligned}$$

Then for each $x, y \in A_n$,

(i) $x \mathcal{D} y$ exactly when $d(x) = d(y)$.
(ii) $x \mathcal{R} y$ exactly when $r(x) = r(y)$.
(iii) $x \mathcal{L} y$ exactly when $l(x) = l(y)$.

We note that $r(x)$ and $l(x)$ correspond to elements of the set $S_n(k)$ of [11], where $k = d(x)$. Now $m(x, y)$ depends only on the first components of $l(x)$ and $r(y)$. If $y \mathcal{R} z$ then $r(y) = r(z)$ by Theorem 17, so that $\alpha(x, y) = \alpha(x, z)$. Consider a $\mathcal{D}$ class D in A_n. Theorem 17 implies that $D = d^{-1}(n-k)$ for some integer k with $0 \leq k \leq n$. Let 1_D denote the element of A_n whose equivalence classes are

$$\tilde{1}_D = \{\{1, 2, \ldots, k\}, \{1', 2', \ldots, k'\}\} \cup \{\{i, i'\} \mid k < i \leq n\}.$$

It is clear that $1_D \in D$ is an idempotent invariant under $*$. Moreover $x \in G_D$ exactly when

$$r(x) = r(1_D) = (\{\{1, 2, \ldots, k\}\}, \{\{i\} \mid k < i \leq n\})$$

and

$$l(x) = l(1_D) = (\{\{1', 2', \ldots, k'\}\}, \{\{i'\} \mid k < i \leq n\})\,.$$

Thus x differs from 1_D only by how the elements $k+1, k+2, \ldots, n$ are paired with the elements $(k+1)', (k+2)', \ldots, n'$. It then follows quickly from the multiplication in A_n that there is a group isomorphism θ_D from the symmetric group S_{n-k} to G_D such that

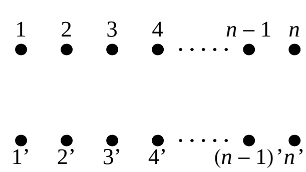
$$\widetilde{\theta_D(\sigma)} = \{\{1, 2, \ldots, k\}, \{1', 2', \ldots, k'\}\} \cup \{\{k + \sigma(i), (k+i)'\} \mid 1 \leq i \leq n-k\}.$$

Moreover $*$ corresponds under θ_D to inversion in S_{n-k}. From example (1.2) of [5], we know that $R[S_{n-k}]$ is cellular with the anti-involution induced by inversion. Therefore $R[G_D]$ is cellular with anti-involution $*$. The assumptions of Corollary 7 are then satisfied, so the partition algebra $R^\alpha[A_n]$ is cellular.

8 The Brauer and Temperley-Lieb Algebras

Suppose Corollary 7 applies to $R^\alpha[S]$, and we wish to apply it to $R^\alpha[T]$, where T is a subsemigroup of S fixed setwise by the involution $*$. Restricting α and $*$ to T, Assumptions 1 and 2 clearly still hold. Moreover if y and z are $\mathcal{R}$ related in T, they are certainly $\mathcal{R}$ related in S, so $\alpha(x, y) = \alpha(x, z)$ for $x \in T$. It therefore suffices to check Assumption 3 and that the relevant group algebras are cellular with anti-involution $*$.

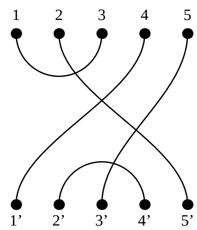
Let BR_n denote the set of elements of A_n whose equivalence classes each contain 2 elements. Thus BR_n essentially consists of all partitions of the set $I \cup I'$ into pairs. We represent elements of BR_n as diagrams by arranging $2n$ dots in the plane and labelling them as shown below, and joining the pairs with arcs.



For example, the element

$$x = \{\{1, 3\}, \{2, 5'\}, \{4, 1'\}, \{5, 3'\}, \{2', 4'\}\}$$

of BR_5 is represented by the diagram

$x =$  .

In fact BR_n forms a subsemigroup of A_n, called the *Brauer semigroup* [12,10]. The twisted semigroup algebra $R^{\alpha}[BR_n]$ is called the *Brauer algebra*. This algebra has been studied extensively in the literature; for example, see [1,7,19]. It was realized as a twisted semigroup algebra as above in [10]. The Green's relations in BR_n are described by the following result, given in Theorem 7 of [12].

Theorem 18. *For $x \in BR_n$, define the functions*

$$\begin{aligned} r(x) &= \{\{i,j\} \in x \mid 1 \le i,j \le n\}, \\ l(x) &= \{\{i',j'\} \in x \mid 1 \le i,j \le n\}, \\ d(x) &= \#\{\{i,j'\} \in x \mid 1 \le i,j \le n\}. \end{aligned}$$

Note that

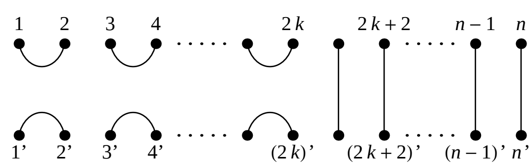
$$d(x) = n - 2|r(x)| = n - 2|l(x)| \in \{n, n-2, n-4, \ldots\}.$$

Suppose $x, y \in BR_n$. Then

(i) $x \,\mathcal{D}\, y$ exactly when $d(x) = d(y)$.
(ii) $x \,\mathcal{R}\, y$ exactly when $r(x) = r(y)$.
(iii) $x \,\mathcal{L}\, y$ exactly when $l(x) = l(y)$.

Now the $\mathcal{D}$ class $D = d^{-1}(n-2k)$ contains the following idempotent.

$$\begin{aligned} 1_D = &\{\{2i-1, 2i\} \mid 1 \le i \le k\} \cup \{\{(2i-1)', (2i)'\} \mid 1 \le i \le k\} \\ &\cup \{\{i, i'\} \mid 2k+1 \le i \le n\} \end{aligned}$$

$=$  .

As in the previous section, 1_D is fixed by $*$ and its $\mathcal{H}$ class is isomorphic to the symmetric group S_{n-2k}, with $*$ corresponding to inversion. By the above discussion, it follows that Corollary 7 applies to the Brauer algebra.

To determine the resulting cell datum, we must choose appropriate elements u_L for each $L \in \mathbb{L}_D$. By Theorem 18, each L is determined uniquely by $l(L)$. Now $l(L)$ consists of k disjoint pairs of elements of the set $\{1', 2', \ldots, n'\}$. Suppose that the remaining $n-2k$ elements are $\{j_1', j_2', \ldots, j_{n-2k}'\}$, where

$$j_1 < j_2 < \ldots < j_{n-2k}.$$

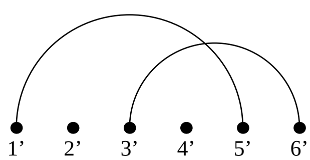
Let

$$u_L = \{\{2i-1, 2i\} \mid 1 \leq i \leq k\} \cup l(L) \cup \{\{2k+i, j_i'\} \mid 1 \leq i \leq n-2k\}.$$

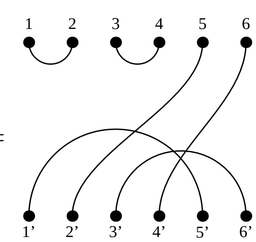
Diagrammatically, $l(L)$ determines the k edges which have both vertices on the bottom row, while $u_L \mathcal{R}\, 1_D$ implies that u_L must contain the k edges

$$r(1_D) = \{\{2i-1, 2i\} \mid 1 \leq i \leq k\}$$

which have both vertices on the top row. The last $n-2k$ dots on the top row are joined to the remaining $n-2k$ dots on the bottom row in the natural way. For example, suppose that $n = 6$ and $k = 2$, and consider the $\mathcal{L}$ class L such that $l(L)$ is represented by

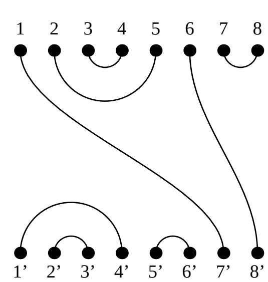


Then

$u_L =$  .

Having thus defined u_L, the cell datum produced by Corollary 7 is exactly that given in [5].

The *Temperley-Lieb semigroup* TL_n is the subsemigroup of BR_n consisting of the diagrams that can be drawn without intersecting curves. For example, an element of TL_8 is shown below.



The twisted semigroup algebra $R^\alpha[TL_n]$ is called the *Temperley-Lieb algebra* [4,9]. Corollary 7 applies to this algebra in the same way. Indeed the $\mathcal{D}$ classes of TL_n correspond to those of BR_n, and the idempotents in BR_n constructed above are contained in TL_n. The maximal groups are trivial in this case, so the group algebras are trivially cellular. Moreover choosing u_L as above, the cell datum produced by Corollary 7 is again the same as in [5].

The cyclotomic Brauer [8] and Temperley-Lieb [17] algebras are variations on the Brauer and Temperley-Lieb algebras which depend on an additional positive integer parameter m. They were shown to be cellular in [18] and [17] respectively, provided the polynomial $x^m - 1$ can be decomposed into linear factors over the ground ring R. Again we can reproduce these results using Corollary 7. Indeed when realising these algebras as twisted semigroup algebras, the underlying semigroups of diagrams have $\mathcal{D}$ classes corresponding to those in BR_n, and idempotents can be chosen analogous to those above. In the case of the cyclotomic Brauer algebra, the maximal subgroups are wreath products $\mathbb{Z}_m \wr S_k$, the group algebra of which is cellular (with the appropriate anti-involution) by Theorem (5.5) of [5]. In the case of the cyclotomic Temperley-Lieb algebra, the maximal subgroups are direct sums of copies of $\mathbb{Z}_m$, the group algebra of which is easily shown to be cellular.

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